geometry." "But suppose the physical space of our experience to be thus only approximately Euclidean space, what is the consequence which follows ?"

The very next year this ever interesting subject recurs in the paper (May 27, 1884) "On the Non-Euclidean Plane Geometry." "Thus the geometry of the pseudo-sphere, using the expression straight line to denote a geodesic of the surface, is the Lobatschewskian geometry ; or rather I would say this in regard to the metrical geometry, or trigonometry, of the surface ; for in regard to the descriptive geometry, the statement requires some qualification . . . this is not identical with the Lobatschewskian geometry, but corresponds to it in a manner such as that in which the geometry of the surface of the circular cylinder corresponds to that of the plane.

I would remark that this realization of the Lobatschewskian geometry sustains the opinion that Euclid's twelfth axiom is undemonstrable."

But here this necessarily brief notice must abruptly stop.

Cayley in addition to his wondrous originality was assuredly the most learned and erudite of mathematicians. Of him in his science it might be said, he knew everything, and he was the very last man who ever will know everything. His was a very gentle, sweet character. Sylvester told me he never saw him angry but once, and that was (both were practicing law !) when a messenger broke in on one of their interviews with a mass of legal documents, new business for Cayley. In an excess of disgust, Cayley dashed the documents upon the floor.

*Austin, Texas, February, 1899.*

# NOTE ON SPHERICAL GEOMETRY.

By G. B. M. ZERR, A. M., Ph. D., Chester, Pa.

DEFINITION. Two arcs of great circles drawn from the vertex of a spherical triangle making equal angles with the spherical bisector of the angle at that vertex are called *isogonal conjugate arcs*. If three arcs drawn through the vertices of a spherical triangle are concurrent, their *isogonal conjugates* with respect to the angles at these vertices are also concurrent.

Let the arcs $AM_a$, $BM_b$, $CM_c$ be concurrent at $M$. To prove that their isogonal conjugates $AK_a$, $BK_b$, $CK_c$ are concurrent.

Fig. 1. Let $BM_a=a_1$, $CM_a=a_2$, $CM_b=b_1$, $AM_b=b_2$, $AM_c=c_1$, $BM_c=c_2$, $BK_a=a_3$, $CK_a=a_4$, $CK_b=b_3$, $AK_b=b_4$, $AK_c=c_3$, $BK_c=c_4$.

$\angle CMM_b=x$, $\angle CMM_a=y$, $\angle BMM_a=z$, $\angle CAM_a=\angle BAK_a=\theta$, $\angle BAM_a=$

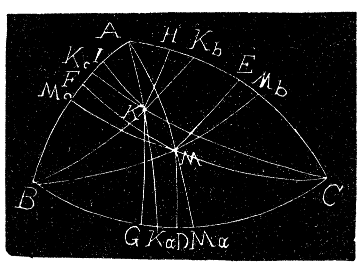


Fig. 1.

$\angle CAK_a = A-\theta$, $\angle CBM_b = \angle ABK_b = \varphi$, $\angle ABM_b = \angle CBK_b = B-\varphi$, $\angle BCM_c = \angle ACK_c = \psi$, $\angle ACM_c = \angle BCK_c = C-\psi$.

Now we have

$$\sin a_1 : \sin BM = \sin z : \sin M_a$$
$$\sin a_2 : \sin CM = \sin y : \sin M_a.$$

$$\therefore \frac{\sin a_1}{\sin a_2} = \frac{\sin BM \sin z}{\sin CM \sin y} \quad \ldots\ldots (1).$$

Similarly, $$\frac{\sin b_1}{\sin b_2} = \frac{\sin CM \sin x}{\sin AM \sin z} \quad \ldots\ldots (2).$$

$$\frac{\sin c_1}{\sin c_2} = \frac{\sin AM \sin y}{\sin BM \sin x} \quad \ldots\ldots (3).$$

Multiplying (1), (2), (3) together we get as the condition of concurrence, the following:

$$\frac{\sin a_1 \sin b_1 \sin c_1}{\sin a_2 \sin b_2 \sin c_2} = 1 \quad \ldots\ldots (4).$$

Now
$$\sin a_3 : \sin AK_a = \sin\theta : \sin B$$
$$\sin a_4 : \sin AK_a = \sin(A-\theta) : \sin C.$$

$$\therefore \frac{\sin a_3}{\sin a_4} = \frac{\sin\theta \sin C}{\sin(A-\theta)\sin B}.$$

But
$$\sin a_1 : \sin AM_a = \sin(A-\theta) : \sin B$$
$$\sin a_2 : \sin AM_a = \sin\theta : \sin C.$$

$$\therefore \frac{\sin\theta}{\sin(A-\theta)} = \frac{\sin a_2 \sin C}{\sin a_1 \sin B}. \qquad \therefore \frac{\sin a_3}{\sin a_4} = \frac{\sin a_2 \sin^2 C}{\sin a_1 \sin^2 B}.$$

Similarly, $$\frac{\sin b_3}{\sin b_4} = \frac{\sin b_2 \sin^2 A}{\sin b_1 \sin^2 C}, \qquad \frac{\sin c_3}{\sin c_4} = \frac{\sin c_2 \sin^2 B}{\sin c_1 \sin^2 A}.$$

$$\therefore \frac{\sin a_3 \sin b_3 \sin c_3}{\sin a_4 \sin b_4 \sin c_4} = \frac{\sin a_2 \sin b_2 \sin c_2}{\sin a_1 \sin b_1 \sin c_1} = 1 \quad \ldots\ldots (5).$$

$\therefore$ $AK_a$, $BK_b$, $CK_c$ are concurrent at $K$.

The two points $M$, $K$ are called *spherical isogonal conjugate points* with respect to the triangle.

Let $MD$, $ME$, $MF$, $KG$, $KH$. $KI$ be the spherical distances (perpendicular arcs) of $M$, $K$ from the sides $a$, $b$, $c$ of the triangle.

Then if $\sin MD : \sin ME : \sin MF = \beta : \gamma : \delta$, it can be demonstrated that $\sin KG : \sin KH : \sin KI = 1/\beta : 1/\gamma : 1/\delta$, as follows:

$$\sin MD = \sin CM \sin\psi,$$
$$\sin ME = \sin CM \sin(C-\psi).$$

$$\therefore \frac{\sin MD}{\sin ME} = \frac{\sin\psi}{\sin(C-\psi)} = \beta/\gamma. \quad \therefore \sin MD : \sin ME = \beta : \gamma.$$

$$\sin KG = \sin CK \sin(C-\psi),$$
$$\sin KH = \sin CK \sin\psi.$$

$$\therefore \frac{\sin KG}{\sin KH} = \frac{\sin(C-\psi)}{\sin\psi} = \gamma/\beta. \quad \therefore \sin KG : \sin KH = \gamma : \beta = 1/\beta : 1/\gamma.$$

In the same way it can be shown that $\sin KH : \sin KI = 1/\gamma : 1/\delta$.

$\therefore \sin KG : \sin KH : \sin KI = 1/\beta : 1/\gamma : 1/\delta$.

DEFINITION. If two arcs of great circles are drawn from the vertex of a spherical triangle cutting the base equally distant from the mid-point, the two arcs thus drawn are called *isotomic conjugate arcs*.

If any three arcs drawn from the vertices $A$, $B$, $C$ of a spherical triangle to the opposite sides are concurrent, their *isotomic* conjugates are also concurrent.

This follows at once from (5) since $a_2 = a_3$, $a_1 = a_4$, $b_2 = b_3$, $b_1 = b_4$, $c_2 = c_3$, $c_1 = c_4$.

The two points thus determined are called *spherical isotomic conjugate points*.

Fig. 2. Let $RD$, $RE$, $RF$, $PG$, $PH$, $PI$ be the spherical distances of $R$, $P$ from the sides $a$, $b$, $c$ of the triangle.

Then if $\sin RD : \sin RE : \sin RF = l : m : r$ it can be demonstrated that $\sin PG : \sin PH : \sin PI = \dfrac{1}{l\sin^2 a} : \dfrac{1}{m\sin^2 b} : \dfrac{1}{n\sin^2 c}$ as follows:

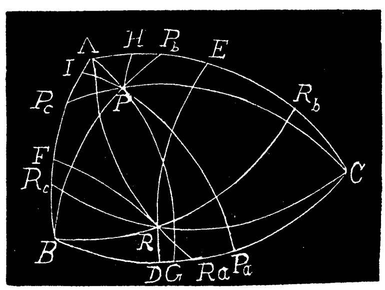

Fig. 2.

Let $BR_a = CP_a = a_1$, $BP_a = CR_a = a_2$,
$CR_b = AP_b = b_1$, $CP_b = AR_b = b_2$,
$BR_c = AP_c = c_1$, $BP_c = AR_c = c_2$.

Then $\sin RD = \sin RC \sin RCD$,
$\sin RE = \sin RC \sin RCE$.

$$\therefore \frac{\sin RD}{\sin RE} = \frac{\sin RCD}{\sin RCE}.$$

But
$$\sin RCD : \sin R_c = \sin c_1 : \sin a$$
$$\sin RCE : \sin R_c = \sin c_2 : \sin b.$$

$$\therefore \frac{\sin RCD}{\sin RCE} = \frac{\sin b \sin c_1}{\sin a \sin c_2}. \quad \therefore \frac{\sin RD}{\sin RE} = \frac{\sin b \sin c_1}{\sin a \sin c_2} = l/m.$$

$$\sin PG = \sin PC \sin PCG$$
$$\sin PH = \sin PC \sin PCH.$$

$$\therefore \frac{\sin PG}{\sin PH} = \frac{\sin PCG}{\sin PCH}.$$

But $\sin PCG : \sin P_c = \sin c_2 : \sin a$

$\sin PCH : \sin P_c = \sin c_1 : \sin b.$

$$\therefore \frac{\sin PCG}{\sin PCH} = \frac{\sin b \sin c_2}{\sin a \sin c_1}. \quad \therefore \frac{\sin PG}{\sin PH} = \frac{\sin b \sin c_2}{\sin a \sin c_1} = \frac{\sin^2 b}{\sin^2 a}.m/l.$$

$$\therefore \sin PG : \sin PH = \sin^2 b.m : \sin^2 a.l \text{ or } \sin PG : \sin PH = \frac{1}{l\sin^2 a} : \frac{1}{m\sin^2 b}.$$

In the same way it can be demonstrated that

$$\sin PH : \sin PI = \frac{1}{m\sin^2 b} : \frac{1}{n\sin^2 c}.$$

$$\therefore \sin PG : \sin PH : \sin PI = \frac{1}{l\sin^2 a} : \frac{1}{m\sin^2 b} : \frac{1}{n\sin^2 c}.$$

EXAMPLES. If $M$ is the median point, $K$ is the symmedian point.

In this case $\dfrac{\sin (C-\psi)}{\sin\psi} = \dfrac{\sin A}{\sin B}$.

$\therefore \sin MD : \sin ME : \sin MF = \operatorname{cosec} A : \operatorname{cosec} B : \operatorname{cosec} C$.

$\sin KG : \sin KH : \sin KI = \sin A : \sin B : \sin C.$

If $R$ is the point of concurrence of arcs drawn from the angles to the points of contact of the incircle with the opposite sides; $P$, its isotomic conjugate point, is the point of concurrence of arcs drawn from the angles to the points of contact of the ex-circles with the opposite sides.

In this case $c_1 = (s-b)$, $c_2 = (s-a)$.

$$\therefore \frac{\sin RD}{\sin RE} = \frac{\sin b \sin(s-b)}{\sin a \sin(s-a)} = \frac{\cos^2 \frac{1}{2}B}{\cos^2 \frac{1}{2}A} = \frac{\sec^2 \frac{1}{2}A}{\sec^2 \frac{1}{2}B}.$$

$$\therefore \sin RD : \sin RE : \sin RF = \text{sic}^2 \tfrac{1}{2}A : \sec^2 \tfrac{1}{2}B : \sec^2 \tfrac{1}{2}C.$$

$$\therefore \sin PG : \sin PH : \sin PI = \frac{\cos^2 \frac{1}{2}A}{\sin^2 a} : \frac{\cos^2 \frac{1}{2}B}{\sin^2 b} : \frac{\cos^2 \frac{1}{2}C}{\sin^2 c}.$$

If we start with the in-center, whose distances from the sides are $1:1:1$, and take its isotomic conjugate we get a point whose distances from the sides are

$$\frac{1}{\sin^2 a} : \frac{1}{\sin^2 b} : \frac{1}{\sin^2 c}.$$

The isogonal conjugate of this last point is a point whose distances from the sides are $\sin^2 a : \sin^2 b : \sin^2 c$. By repeating this process we get a series of points whose distances from the sides are as $\sin^m a : \sin^m b : \sin^m c$, where $m$ is an even positive or negative integer. For the symmedian point we have $\sin A : \sin B : \sin C = \sin a : \sin b : \sin c$. Starting with this point and alternating as above, we get a series of points whose distances from sides are as $\sin^n a : \sin^n b : \sin^n c$ where $n$ is any odd positive or negative integer.

## NEW AND OLD PROOFS OF THE PYTHAGOREAN THEOREM.

By BENJAMIN F. YANNEY, A. M., Mount Union College, Alliance, Ohio, and JAMES A. CALDERHEAD, B. Sc., Curry University, Pittsburg, Pennsylvania.

[Continued from February Number.]

XCI. Fig. 38 and Fig. 39.

The triangle is $ABC$. The construction in the two figures is evident. The Hindu Bhaskara, the author of this method, complimented his readers by condensing his proof into the single word, "Behold." We follow his example.

NOTE. The above is a conjectured proof of Pythagoras. See pages 50 and 123 of Cajori's "History of Elementary Mathematics."

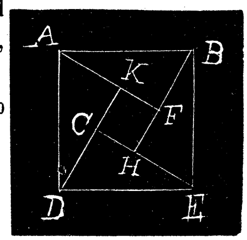

Fig. 38.

XCII. Fig. 38.

$$\begin{aligned}AB^2 &= 2AC.BC + CH^2\\ &= 2AC.BC + (AH - AC)^2\\ &= 2AC.BC + (BC - AC)^2\\ &= BC^2 + AC^2.\end{aligned}$$

Q. E. E.

XCIII. Fig. 38.

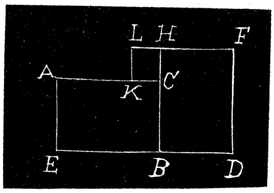

Fig. 39.

Suppose $BC$ is produced to meet $AD$, as at $L$. Then let $LBA$ be the given triangle, right-angled at $A$.

Now, the the area of the square on $AB$=the sum of the four triangles $ABC$, $BEK$, $DEF$ and $ADH$, and the square $CHFK$; or, $AB^2 = 2AC.BC + CH^2$.

Again, $AC = (AL.AB) \div BL$, $BC = AB^2 \div BL$, and $CH = AH - AC = BC - AC$.

$$\therefore\ 2AC.BC + CH^2 = BC^2 + AC^2 = \frac{AB^4}{BL^2} + \frac{AL^2.AB^2}{BL^2}.$$

$$\therefore\ BL^2 = AB^2 + AL^2.$$

Q. E. D.